Stability-Preserving Rational Approximation Subject to Interpolation Constraints

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***Abstract*—A quite comprehensive theory of analytic interpolation with degree constraint, dealing with rational analytic interpolants with an a priori bound, has been developed in recent years. In this paper we consider the limit case when this bound is removed, and only stable interpolants with a prescribed maximum degree are sought. This leads to weighted H_2 minimization, where the interpolants are parameterized by the weights. The inverse problem of determining the weight given a desired interpolant profile is considered, and a rational approximation procedure based on the theory is proposed. This provides a tool for tuning the solution to specifications. The basic idea could also be applied to the case with bounded analytic interpolants.**

I. Introduction

STABILITY-PRESERVING model reduction is a topic of major importance in systems and control, and over the last decades numerous such approximation procedures have been developed; see, e.g., [5], [18], [27], [2] and references therein. In this paper we introduce a novel approach to stability- preserving model reduction that also accommodates interpolation constraints, a requirement not uncommon in systems and control. By choosing the weights appropriately in a family of weighted H_2 minimization problems, the minimizer will both have low degree and match the original system.

As we shall see in this paper, stable interpolation with degree constraint can be regarded as a limit case of bounded analytic interpolation under the same degree constraint – a topic that has been thoroughly researched in recent years; see [7], [11].

More precisely, let f be a function in $H(\mathbb{D})$, the space of functions analytic in the unit disc $\mathbb{D} = \{z : |z| < 1\}$, satisfying

(i) the interpolation condition

$$f(z_k) = w_k, \quad k = 0, \dots, n, \tag{1}$$

(ii) the a priori bound $\|f\|_\infty \le \gamma$, and

(iii) the condition that f be rational of degree at most n,

where $z_0, z_1, \dots, z_n \in \mathbb{D}$ are distinct (for simplicity) and $w_0, w_1, \dots, w_n \in \mathbb{C}$. It was shown in [7] that, for each such f, there is a rational function $\sigma(z)$ of the form

$$\sigma(z) = \frac{p(z)}{\tau(z)}, \qquad \tau(z) := \prod_{k=0}^{n}(1 - \bar{z}_k z),$$

where $p(z)$ is a polynomial of degree n with $p(0) > 0$ and $p(z) \neq 0$ for $z \in \mathbb{D}$ such that f is the unique minimizer of the generalized entropy functional

$$-\int_{-\pi}^{\pi} |\sigma(e^{i\theta})|^2 \gamma^2 \log(1 - \gamma^{-2}|f(e^{i\theta})|^2) \frac{d\theta}{2\pi}$$

subject to the interpolation conditions (1). In fact, there is a complete parameterization of the class of all interpolants satisfying (i)–(iii) in terms of the zeros of σ, which also are the spectral zeros of f; i.e., the zeros of $\gamma^2 - f(z)f^*(z)$ located in the complement of the unit disc. It can also be shown that this parameterization is smooth, in fact a diffeomorphism [8].

This smooth parameterization in terms of spectral zeros is the center piece in the theory of analytic interpolation with degree constraints; see [6], [7] and reference therein. By tuning the spectral zeros one can obtain an interpolant that better fulfills additional design specifications. However, one of the stumbling-blocks in the application of this theory has been the lack of a systematic procedure for achieving this tuning. In fact, the relation between the spectral zeros of f and f itself is nontrivial, and how to choose the spectral zeros in order to obtain an interpolant which satisfy the given design specifications is a partly open problem.

In order to understand this problem better, we will in this paper focus on the limit case as $\gamma \to \infty$; i.e., the case when condition (ii) is removed. We shall refer to this problem – which is of considerable interest in its own right – as *stable interpolation with degree constraint*. Note that, as $\gamma \to \infty$,

$$-\gamma^2 \log(1 - \gamma^{-2}|f|^2) \to |f|^2,$$

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and hence (see Proposition 2),

$$-\int_{-\pi}^{\pi} |\sigma|^2\gamma^2 \log(1-\gamma^{-2}|f|^2)\frac{d\theta}{2\pi} \to \int_{-\pi}^{\pi} |\sigma f|^2 \frac{d\theta}{2\pi}.$$

For the case $\sigma \equiv 1$, this connection between the H_2 norm and the corresponding entropy functional have been studied in [16]. Consequently, the stable interpolants with degree constraint turn out to be minimizers of weighted H_2 norms. Indeed, the H_2 norm plays the same role in stable interpolation as the entropy functional does in bounded interpolation. Stable interpolation and H_2 norms are considerably easier to work with than bounded analytic interpolation and entropy functionals, but many of the concepts and ideas are similar.

The purpose of this paper is twofold. First, we want to provide a stability-preserving model reduction procedure that admits interpolation constraints and error bounds. Secondly, this theory is the simplest and most transparent gateway for understanding the full power of bounded analytic interpolation with degree constraint. In fact, our paper provides, together with the results in [12], the key to the problem of how to settle an important open question in the theory of bounded analytic interpolation with degree constraint, namely how to choose spectral zeros.

In many applications, no interpolation conditions are given a priori. This allows us to use the interpolation points as additional tuning variables, available for satisfying design specifications. Such an approach for passivity-preserving model reduction was proposed in [1] and further developed in [20]. In [10] we pointed out that the procedure in [1], [20] can be regarded as a special case of Nevanlinna-Pick interpolation with degree constraint [6], namely the central solution corresponding to the choice $\sigma \equiv 1$. We demonstrated in [10] that using the full power of the latter theory allows us the tune the approximant to specifications without increasing the degree. A similar situation occurs in the context of prediction-error approximation with a finite set of basis functions [13], [23], which together with prefiltering leads to the minimization of generalized entropy functionals [3].

A problem left open in [10] and in [13] was how to actually select spectral zeros and interpolation points in a systematic way in order to obtain the best approximation. It is precisely this problem, here in the context of stability-preserving model reduction, that is the topic of this paper. Unlike the procedures in [1], [20], [10], we shall also be able to provide error bounds. Moreover, although the procedures presented in this paper are in the setting of stable interpolation, they will also give insight into both bounded analytic interpolation [7] and positive real interpolation [6].

The paper is outlined as follows. In Section II we show that the problem of stable interpolation is the limit, as the bound tend to infinity, of the bounded analytic interpolation problem stated above. In Section III we derive the basic theory for how all stable interpolants with a degree bound may be obtained as weighted H_2-norm minimizers. In Section IV we consider the inverse problem of H_2 minimization, and in Section V the inverse problem is used for model reduction of interpolants. The inverse problem and the model reduction procedure are closely related to the theory in [12]. A model reduction procedure where no a priori interpolation conditions are required are derived in Section VI. This is motivated by a weighed relative error bound of the approximant and gives a systematic way to choose the interpolation points. This approximation procedure is also tunable so as to give small error in selected regions. In the Appendix we describe how the corresponding quasi-convex optimization problems can be solved. Finally, in Section VII we illustrate our new approximation procedures by applying them to a simple example and conclude with a control design example.

II. Bounded Interpolation and Stable Interpolation

In this section we show that the H_2 norm is the limit of a sequence of entropy functionals. From this limit, the relation between stable interpolation and bounded interpolation is established, and it is shown that some of the important concepts in the two different frameworks match.

First consider one of the main results of bounded interpolation: a complete parameterization of all interpolants with a degree bound [7]. For this, we will need two key concepts in that theory; the entropy functional

$$\mathbb{K}^{\gamma}_{|\sigma|^2}(f) = -\int_{-\pi}^{\pi} \gamma^2 |\sigma(e^{i\theta})|^2 \log(1-\gamma^{-2}|f(e^{i\theta})|^2)\frac{d\theta}{2\pi},$$

where we take $\mathbb{K}^{\gamma}_{|\sigma|^2}(f) := \infty$ whenever the H_∞ norm $\|f\|_\infty > \gamma$, and the co-invariant subspace

$$\mathcal{K} = \left\{ \frac{p(z)}{\tau(z)} \,:\, \tau(z) = \prod_{k=0}^{n}(1-\bar{z}_k z), p \in \mathrm{Pol}(n) \right\}. \tag{2}$$

Here $\mathrm{Pol}(n)$ denotes the set of polynomials of degree at most n, and $\{z_k\}_{k=0}^n$ are the interpolation points.

In fact, any interpolant f of degree at most n with $\|f\|_\infty \le \gamma$ is a minimizer of $\mathbb{K}^{\gamma}_{|\sigma|^2}(f)$ subject to (1) for some $\sigma \in \mathcal{K}_0$, where

$$\mathcal{K}_0 = \{\sigma \in \mathcal{K} \,:\, \sigma(0) > 0, \sigma \text{ outer}\}.$$

Furthermore, all such interpolants are parameterized by $\sigma \in \mathcal{K}_0$. This is one of the main results for bounded interpolation in [7] and is stated more precisely as follows.

Theorem 1: Let $\{z_k\}_{k=0}^n \subset \mathbb{D}, \{w_k\}_{k=0}^n \subset \mathbb{C}$, and $\gamma \in \mathbb{R}_+$. Suppose that the Pick matrix

$$P = \left[\frac{\gamma^2 - w_k \bar{w}_\ell}{1 - z_k \bar{z}_\ell}\right]_{k,\ell=0}^n \tag{3}$$

is positive definite, and let σ be an arbitrary function in $\mathcal{K}_0$. Then there exists a unique pair of elements $(a, b) \in \mathcal{K}_0 \times \mathcal{K}$ such that

(i) $f(z) = b(z)/a(z) \in H^\infty$ with $\|f\|_\infty \le \gamma$
(ii) $f(z_k) = w_k, \quad k = 0, 1, \dots, n$, and
(iii) $|a(z)|^2 - \gamma^{-2}|b(z)|^2 = |\sigma(z)|^2$ for $z \in \mathbb{T}$,

where $\mathbb{T} := \{z : |z| = 1\}$. Conversely, any pair $(a, b) \in \mathcal{K}_0 \times \mathcal{K}$ satisfying (i) and (ii) determines, via (iii), a unique $\sigma \in \mathcal{K}_0$. Moreover, the optimization problem

$$\min \mathbb{K}^\gamma_{|\sigma|^2}(f) \ \text{ s.t. } \ f(z_k) = w_k, \ k = 0, \dots, n$$

has a unique solution f that is precisely the unique f satisfying conditions (i), (ii) and (iii).

The essential content of this theorem is that the class of interpolants satisfying $\|f\|_\infty \le \gamma$ may be parameterized in terms of the zeros of σ, and that these zeros are the same as the *spectral zeros* of f; i.e., the zeros of the spectral outer factor $w(z)$ of $w(z)w^*(z) = \gamma^2 - f(z)f^*(z)$, where $f^*(z) = \overline{f(\bar{z}^{-1})}$.

Let $\|f\| = \sqrt{< f, f >}$ denote the norm in the Hilbert space $H_2(\mathbb{D})$ with inner product

$$< f, g >= \int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\frac{d\theta}{2\pi},$$

and let $RH(\mathbb{D})$ denote the rational functions analytic in $\mathbb{D}$. As the bound γ tend to infinity,

$$-\gamma^2 \log(1 - \gamma^{-2}|f|) \to |f|^2.$$

Therefore, the entropy functional $\mathbb{K}^\gamma_{|\sigma|^2}(f)$ converge to the weighted H_2 norm $\|\sigma f\|^2$.

Proposition 2: Let $f, \sigma \in RH(\mathbb{D})$ with σ outer and $\|f\|_\infty < \infty$. Then

(i) $\mathbb{K}^\gamma_{|\sigma|^2}(f)$ is a non-increasing function of γ, and,
(ii) $\mathbb{K}^\gamma_{|\sigma|^2}(f) \to \|\sigma f\|^2$ as $\gamma \to \infty$.

Proof: It clearly suffices to consider only $\gamma \ge \|f\|_\infty$. Then the derivative of $-\gamma^2 \log(1-\gamma^{-2}|f|^2)$ with respect to γ is non-positive for $|f| \le \gamma$, and hence $\mathbb{K}^\gamma_{|\sigma|^2}(f)$ is non-increasing. To establish (ii), note that

$$-\gamma^2 \log(1 - \gamma^{-2}|f|^2) = |f|^2 + O(\gamma^{-2}|f|^2),$$

and therefore

$$-|\sigma|^2\gamma^2 \log(1 - \gamma^{-2}|f|^2) \to |\sigma f|^2$$

pointwise in $\mathbb{T}$ except for σ with poles in $\mathbb{T}$. There are two cases of importance. First, if σ has no poles in $\mathbb{T}$, or if a pole of σ coincided with a zero of f of at least the same multiplicity, then $-|\sigma|^2\gamma^2 \log(1 - \gamma^{-2}|f|^2)$ is bounded, and (ii) follows from bounded convergence. Secondly, if σ has a pole in $\mathbb{T}$ at a point in which f does not have a zero, then both $\mathbb{K}^\gamma_{|\sigma|^2}(f)$, and $\|\sigma f\|^2$ are infinite for any γ. ∎

The condition $\|f\|_\infty < \infty$ is needed in Proposition 2. Otherwise, if $\|f\|_\infty = \infty$, then $\mathbb{K}^\gamma_{|\sigma|^2}(f)$ is infinite for any γ, while $\|\sigma f\|^2$ may be finite if σ has zeros in the poles of f on $\mathbb{T}$.

The next proposition shows that stable interpolation may be seen as the limit case of bounded interpolation when the bound γ tend to infinity.

Proposition 3: Let σ be any outer function such that the minimizer f of

$$\min \|\sigma f\| \text{ such that } f(z_k) = w_k, k = 0, \dots, n \tag{4}$$

satisfies $\|f\|_\infty < \infty$. Let f_γ be the minimizer of

$$\min \mathbb{K}^\gamma_{|\sigma|^2}(f_\gamma) \text{ such that } f_\gamma(z_k) = w_k, k = 0, \dots, n$$

for $\gamma \in \mathbb{R}_+$ large enough so that the Pick matrix (3) is positive definite. Then $\|\sigma(f - f_\gamma)\| \to 0$ as $\gamma \to \infty$.

Proof: By Proposition 2, and since f and f_γ are minimizers of the respective functional, we have

$$\mathbb{K}^\gamma_{|\sigma|^2}(f) \ge \mathbb{K}^\gamma_{|\sigma|^2}(f_\gamma) \ge \|\sigma f_\gamma\|^2 \ge \|\sigma f\|^2.$$

Moreover, since $\mathbb{K}^\gamma_{|\sigma|^2}(f) \to \|\sigma f\|^2$ as $\gamma \to \infty$ it follows that $\|\sigma f_\gamma\|^2 \to \|\sigma f\|^2$, and hence, by Lemma 8, we have $\|\sigma(f - f_\gamma)\| \to 0$ as $\gamma \to \infty$, as claimed. ∎

Note that Proposition 3 holds for any σ which is outer and not only for $\sigma \in \mathcal{K}_0$. However, if $\sigma \in \mathcal{K}_0$, then $\deg f_\gamma \le n$ for any γ. Therefore, since $\|\sigma(f - f_\gamma)\| \to 0$ as $\gamma \to \infty$, for $\sigma \in \mathcal{K}_0$ the minimizer f of (4) will be a stable interpolant of degree at most n. We will return to this in the next section.

It is interesting to note how concepts in the two types of interpolation are related. First of all, the weighted H_2 norm plays the same role in stable interpolation as the entropy functional does in bounded interpolation. Secondly, the spectral zeros, which play an major role in degree constrained bounded interpolation, simply correspond to the poles in stable interpolation. This may be seen from (iii) in Theorem 1.

III. Rational Interpolation and H_2 minimization

In the previous section we have seen that minimizers of a specific class of H_2 norms are stable interpolants of degree at most n. This, and also the fact that this class may be parameterized by $\sigma \in \mathcal{K}_0$ can be proved using basic Hilbert space concepts. This will be done in this section.

To this end, first consider the minimization problem

$$\min \|f\| \text{ s. t. } f(z_k) = w_k, k = 0, \ldots, n, \tag{5}$$

without any weight σ. Let $f_0 \in H_2(\mathbb{D})$ satisfy the interpolation condition (1). Then any $f \in H_2(\mathbb{D})$ satisfying (1) can be written as $f = f_0 + v$, where $B = \prod_{k=0}^{n} \frac{z_k - z}{1 - \bar{z_k} z}$ and $v \in BH_2$. Therefore, (5) is equivalent to

$$\min_{v \in BH_2} \|f_0 + v\|.$$

By the Projection Theorem (see, e.g., [14]), there exists a unique solution $f = f_0 + v$ to this optimization problem, which is orthogonal to BH_2, i.e. $f \in \mathcal{K} := H_2 \ominus BH_2$.

Conversely, if $f \in \mathcal{K}$ and $f(z_k) = w_k$, for $k = 0, \ldots, n$, then f is the unique solution of (5). To see this, note that any interpolant in $H_2(\mathbb{D})$ may be written as $f + v$ where $v \in BH_2$. However, since $v \in BH_2 \perp \mathcal{K} \ni f$, we have $\|f + v\|^2 = \|f\|^2 + \|v\|^2$, and hence the minimizer is f, obtained by setting $v = 0$.

We summarize this in the following proposition.

Proposition 4: The unique minimizer of (5) belongs to $\mathcal{K}$. Conversely, if $f \in \mathcal{K}$ and $f(z_k) = w_k$, for $k = 0, \ldots, n$, then f is the minimizer of (5).

Consequently, in view of (2), f is a rational function with its poles fixed in the mirror images (with respect to the unit circle) of the interpolation points. By introducing weighted norms, any interpolant with poles in prespecified points may be constructed in a similar way. In fact, the set of interpolants f of degree $\leq n$ may be parameterized in this way. One way to see this is by considering

$$\min \|\sigma f\| \text{ s. t. } f(z_k) = w_k, k = 0, \ldots, n, \tag{6}$$

where $\sigma \in \mathcal{K}_0$. Since σ is invertible in $H(\mathbb{D})$, (6) is equivalent to

$$\min \|\sigma f\| \text{ s. t. } (\sigma f)(z_k) = \sigma(z_k) w_k, k = 0, \ldots, n.$$

According to Proposition 4, this has the optimal solution $\sigma f = b \in \mathcal{K}$, and hence the solution of (6), $f = \frac{b}{\sigma}$, is rational of degree at most n. To see that any solution of degree at most n can be obtained in this way, note that any such interpolant f is of the form $f = \frac{b}{\sigma}, b \in \mathcal{K}, \sigma \in \mathcal{K}_0$. Since $\sigma f = b \in \mathcal{K}$ holds together with the interpolation condition (1) if and only if $\sigma(z_k) f(z_k) = \sigma(z_k) w_k$ for $k = 0, \ldots, n$, f is the unique solution of (6), by Proposition 4. This proves the following proposition.

Theorem 5: Let $\sigma \in \mathcal{K}_0$. Then the unique minimizer of

$$\min \|\sigma f\| \text{ s. t. } f(z_k) = w_k,\ k = 0, \ldots, n, \tag{7}$$

belong to $H(\mathbb{D})$ and is rational of a degree at most n. More precisely,

$$f = \frac{b}{\sigma} \tag{8}$$

where $b \in \mathcal{K}$ is the unique solution of the linear system of equations

$$b(z_k) = \sigma(z_k) w_k, \quad k = 0, 1, \ldots, n. \tag{9}$$

Conversely, if f satisfies (8) for some $b \in \mathcal{K}$ and the interpolation condition (1), then f is the unique minimizer of (7).

In other words, the set of interpolants in $H(\mathbb{D})$ of degree at most n may be parameterized in terms of weights $\sigma \in \mathcal{K}_0$. Another way to look at this is that the poles of the minimizer (8) are specified by the zeros of σ and that the numerator $b = \beta/\tau$ is determined from the interpolation condition by solving the linear system of equations

$$\beta(z_k) = \tau(z_k) \sigma(z_k) w_k, \quad k = 0, 1, \ldots, n \tag{10}$$

for the $n+1$ coefficients $\beta_0, \beta_1, \ldots, \beta_n$ of the polynomial $\beta(z)$. This is a Vandermonde system that is known to have a unique solution (as long as the interpolation point $z_o, z_1, \ldots, z_n$ are distinct as here).

Note that this parameterization is not necessarily injective. If, for example, $w_k = 1$ for $k = 0, \ldots, n$, then there is a unique function f of degree at most n that satisfies $f(z_k) = w_k, k = 0, \ldots, n$. No matter how $\sigma \in \mathcal{K}_0$ is chosen, $b = \sigma$, and hence the minimizer of (6) will be $f \equiv 1$.

IV. The Inverse Problem

In [12] we considered the *inverse problem of analytic interpolation*; i.e., the problem of choosing an entropy functional whose unique minimizer is a prespecified interpolant. In this section we will consider the counterpart of this problem for stable interpolation.

Suppose $f \in RH(\mathbb{D})$ satisfies the interpolation condition (1). Then, when does there exist $\sigma \in RH(\mathbb{D})$ which is outer such that f is the minimizer of

$$\min \|\sigma f\| \text{ s. t. } f(z_k) = w_k, k = 0, \ldots, n?$$

We refer to this as the *inverse problem of H_2 minimization*, and its solution is given in the following proposition.

Theorem 6: Let $f \in RH(\mathbb{D})$ satisfy the interpolation condition $f(z_k) = w_k, k = 0, \ldots, n$. Then f is the minimizer of

$$\min \|\sigma f\| \text{ s. t. } f(z_k) = w_k,\ k = 0, \ldots, n, \tag{11}$$

where σ is outer if and only if $\sigma f \in \mathcal{K}$, in which case the minimizer is unique. Such a σ exists if and only if f has no more than n zeros in $\mathbb{D}$.

Proof: The function f is the minimizer of (11) if and only if $b = \sigma f$ is the minimizer (necessarily unique) of

$$\min \|b\| \text{ s. t. } b(z_k) := w_k \sigma(z_k),\ k = 0, \ldots, n,$$

which, by Proposition 4, holds if and only if $\sigma f = b \in \mathcal{K}$. Such a σ only exists if f has less or equal to n zeros inside $\mathbb{D}$. To see this, first note that, if f has more than n zeros in $\mathbb{D}$, then σf has more than n zeros in $\mathbb{D}$ and can therefore not be of the form p/τ with $p \in \mathrm{Pol}(n)$. On the other hand, if f has less or equal to n zeros in $\mathbb{D}$, then let $p = \prod(z - p_k)$ where p_k are the zeros of f, and set $\sigma := \frac{p}{f\tau}$. Then σ is outer and satisfies $\sigma f \in \mathcal{K}$. ■

Theorem 6 defines a map F that sends σ to the unique minimizer f of the optimization problem (11); i.e.,

$$\sigma \mapsto f = F(\sigma). \tag{12}$$

Let W_f denote the set of weights σ that give f as a minimizer of (11); i.e., the inverse image $F^{-1}(f)$ of f. By Theorem 6,

$$\begin{aligned} W_f &:= F^{-1}(f) = \{\sigma \text{ outer} : \sigma f \in \mathcal{K}\} \qquad (13)\\ &= \left\{\sigma = \frac{p}{f\tau} : p \in \mathrm{Pol}(n) \setminus \{0\}, \frac{p}{f} \text{ outer}\right\}, \end{aligned}$$

i.e., W_f may be parameterized in terms of the polynomials $p \in \mathrm{Pol}(n)$. For the condition that pf^{-1} is outer to hold for some $p \in \mathrm{Pol}(n)$, it is necessary that f has at most n zeros in $\mathbb{D}$. This is in accordance with Theorem 6. It is interesting to note that the dimension of W_f depends on the number of zeros of f inside $\mathbb{D}$. The more zeros f has inside $\mathbb{D}$, the more restricted is the class W_f. One extreme case is when f has no zeros inside $\mathbb{D}$. Then p could be any stable polynomial of degree n. The other extreme is when f has n zeros in $\mathbb{D}$, in which case p is uniquely determined up to a multiplicative constant.

V. Rational Approximation with Interpolation Constraints

In this section the solution of the inverse problem (Theorem 6) will be used to develop an approximation procedure for interpolants. Let $f \in RH(\mathbb{D})$ be a function satisfying the interpolation condition (1). We want to construct another function $g \in RH(\mathbb{D})$ of degree at most n satisfying the same interpolation condition such that g is as close as possible to f.

Let $\sigma \in W_f$; i.e., let σ be a weight and such that f is the minimizer of (11), and let ρ be close to σ. Then it seems reasonable that the minimizer g of the optimization problem

$$\min \|\rho g\| \text{ s. t. } g(z_k) = w_k,\ k = 0, \ldots, n, \tag{14}$$

is close to f. This is the statement of the following theorem.

Theorem 7: Let $f \in RH(\mathbb{D})$ satisfy the interpolation condition $f(z_k) = w_k, k = 0, \ldots, n$, and let $\sigma \in W_f$. Moreover, let ρ be an outer function such that

$$\left\| 1 - \left| \frac{\rho}{\sigma} \right|^2 \right\|_\infty = \epsilon, \tag{15}$$

and let g be the corresponding minimizer of (14). Then

$$\|\sigma(f - g)\|^2 \leq \frac{4\epsilon}{1 - \epsilon} \|\sigma f\|^2. \tag{16}$$

For the proof we need the following useful lemma.

Lemma 8: Let let $g \in RH(\mathbb{D})$ satisfy $g(z_k) = w_k$ for $k = 0, \ldots, n$, and let f be the minimizer of (11). Then, if

$$\|\sigma g\|^2 \leq (1 + \delta)\|\sigma f\|^2,$$

we have

$$\|\sigma(f - g)\|^2 \leq 2\delta\|\sigma f\|^2.$$

Proof: From the parallelogram law we have,

$$\frac{1}{2}\left(\|\sigma f\|^2 + \|\sigma g\|^2\right) = \left\|\sigma \frac{f + g}{2}\right\|^2 + \left\|\sigma \frac{f - g}{2}\right\|^2.$$

Therefore, since f is the minimizer of (11), and hence $\|\sigma f\| \leq \|\sigma(f + g)/2\|$, it follows that

$$\|\sigma(f - g)\|^2 \leq 2(\|\sigma g\|^2 - \|\sigma f\|^2) \leq 2\delta\|\sigma f\|^2,$$

which concludes the proof of the lemma. ■

Proof of Theorem 7: In view of (15) we have

$$(1 - \epsilon)|\sigma(e^{i\theta})|^2 \leq |\rho(e^{i\theta})|^2 \leq (1 + \epsilon)|\sigma(e^{i\theta})|^2$$

for all $\theta \in [-\pi, \pi]$. Therefore, since g is the minimizer of (14), by (15), we have

$$\begin{aligned} \|\sigma g\|^2 &\leq \frac{1}{1 - \epsilon}\|\rho g\|^2 \leq \frac{1}{1 - \epsilon}\|\rho f\|^2 \\ &\leq \frac{1 + \epsilon}{1 - \epsilon}\|\sigma f\|^2 = (1 + \delta)\|\sigma f\|^2, \end{aligned}$$

where $\delta := 2\epsilon/(1 - \epsilon)$. Consequently (16) follows from Lemma 8. ■

We have thus shown that if $\left|\frac{\rho(z)}{\sigma(z)}\right|$ is close to 1 for $z \in \mathbb{T}$, then $\|\sigma(f - g)\|$ is small. This suggests the following approximation procedure, illustrated in Figure 1. By Theorem 5, the function F, defined by (12), maps the subset $\mathcal{K}_0$ into the space of interpolants of degree at most n. In Figure 1 these subsets are depicted by fat lines. The basic idea is to replace the hard problem

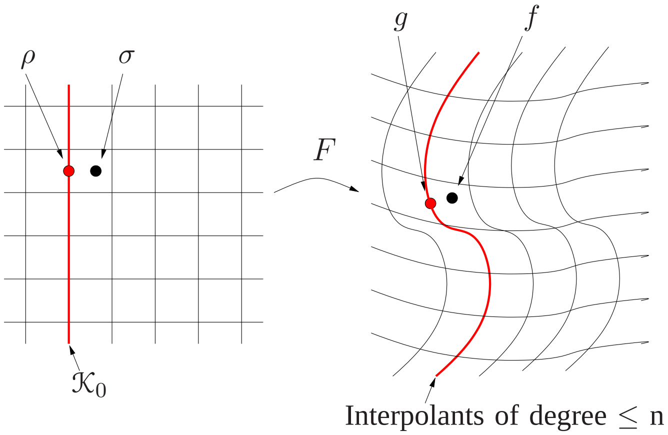

Fig. 1. *The* map F sending weighting functions to interpolants.

of approximating f by a function g of degree at most n by the simpler problem of approximating an outer function σ by a function $\rho \in \mathcal{K}_0$.

Theorem 7 suggests various strategies for choosing the functions $\rho \in \mathcal{K}_0$ and $\sigma \in W_f$ depending on the design preferences. If a small error bound for $\|\sigma(f-g)\|$ is desired for a particular $\sigma \in W_f$, this σ should be used together with the $\rho \in \mathcal{K}_0$ that minimizes (15).

However, obtaining a small value of (15) is often more important than the choice of σ. Therefore, in general it is more natural to choose the pair $(\sigma, \rho) \in (W_f, \mathcal{K}_0)$ that minimizes ϵ. For such a pair, setting $q := \tau\rho$, we can be see from (2) and (13) that

$$\epsilon = \left\| 1 - \left| \frac{\rho}{\sigma} \right|^2 \right\|_\infty = \left\| 1 - \left| \frac{qf}{p} \right|^2 \right\|_\infty, \tag{17}$$

where $q \in \mathrm{Pol}(n)$ and $p \in \mathrm{Pol}(n) \setminus \{0\}$ needs to be chosen so that p/f is outer. It is interesting to note that (17) is independent of $\tau(z) := \prod_{k=0}^{n}(1-\bar{z}_k z)$ and hence of the interpolation points $z_0, z_1, \ldots, z_n$.

Now suppose that f has ν zeros in $\mathbb{D}$; i.e., ν nonminimum-phase zeros. Then $f = \pi f_0$, where f_0 is outer (minimum phase) and π is an unstable polynomial of degree $\nu \leq n$. Setting $p = \pi p_0$, our optimization problem to minimize ϵ reduces to the problem to find a pair $(p_0, q) \in \mathrm{Pol}(n-\nu) \times \mathrm{Pol}(n)$ that minimizes

$$\epsilon = \left\| 1 - \left| \frac{qf_0}{p_0} \right|^2 \right\|_\infty \tag{18}$$

for a given nonminimum-phase f_0. This is a quasi-convex optimization problem, which can be solved as described in the Appendix (see also [21], [24]). The optimal q yields the optimal $\rho = q/\tau$. The approximant g is then obtained by solving the optimization problem (14) as described in Theorem 5.

One should note that, the more zeros f has inside $\mathbb{D}$, the smaller is the choice of p. Therefore one expects approximations of non-minimum phase plants to be worse than approximations of plants without unstable zeros.

VI. Rational Approximation

In applications where there are no a priori interpolation constraints, the choice of interpolation points serve as additional design parameters. It is then important to choose them so that a good approximation is obtained. There are some general guidelines that one could use for manual tuning. The main strategy previously used is to chose interpolation points close to the regions of the unit circle where good fit is desired. The closer to the unit circle the points are placed, the better fit, but the smaller is the region where good fit is ensured; see [10] for further discussions on this. However, in this paper we shall provide a systematic procedure for choosing the interpolation points, based on quasi-convex optimization.

As we have seen in the previous section the choice of interpolation points does not affect ϵ given by (17). However, since $\sigma = \frac{p}{f\tau}$, the weighted H_2 error bound (16) in Theorem 7 becomes

$$\left\| \frac{p}{\tau} \frac{f-g}{f} \right\|^2 \leq \frac{4\epsilon}{1-\epsilon} \left\| \frac{p}{\tau} \right\|^2,$$

which depends on τ and hence on the choice of interpolation points. In fact, this is a weighed H_2 bound on the relative error $(f-g)/f$. If a specific part of the unit circle is of particular interest, interpolation points may be placed close to that part, which gives a bound on the weighted relative error with high emphasis on that specific region. (For a method to do this by convex optimization, see Remark 2 in the next section.) If no particular part is more important than the rest, we suggest to select τ as the outer part of p; i.e., $|\tau(z)| = |p(z)|$ for $z \in \mathbb{T}$. This gives a natural choice of interpolation points that are the mirror images of the roots of τ. Furthermore, this choice gives the relative error bound $\|(f-g)/f\| \leq 4\epsilon/(1-\epsilon)$. This is summarized in the following theorem.

Theorem 9: Let p and q be polynomials of degrees at most n such that pf^{-1} is outer, and set

$$\epsilon := \left\| 1 - \left| \frac{qf}{p} \right|^2 \right\|_\infty. \tag{19}$$

Let $z_0, z_1, \ldots, z_n \in \mathbb{D}$ and let

$$g = \arg\min \|\rho g\| \text{ s. t. } g(z_k) = f(z_k), k = 0, \ldots, n,$$

where $\rho = q/\tau$ and $\tau = \prod_{k=0}^{n}(1-\bar{z}_k z)$. Then

$$\left\|\frac{p}{\tau}\frac{f-g}{f}\right\|^2 \leq \frac{4\epsilon}{1-\epsilon}\left\|\frac{p}{\tau}\right\|^2. \tag{20}$$

In particular, if the interpolation points $z_0, z_1, \ldots, z_n$ are chosen so that $|\tau(z)| = |p(z)|$ for $z \in \mathbb{T}$, then

$$\left\|\frac{f-g}{f}\right\|^2 \leq \frac{4\epsilon}{1-\epsilon}. \tag{21}$$

Remark 1: Note that the choice $|\tau| = |p|$ in Theorem 9 implies that the unstable zeros of f become interpolation points. Therefore, for $\epsilon < 1$, $(f-g)/f$ belongs to H_2.

VII. The Computational Procedure and Some illustrative Examples

Next we summarize the computational procedure suggested by the theory presented above and apply it to some examples.

Given a function $f \in RH(\mathbb{D})$ with at most n zeros in $\mathbb{D}$, we want to construct a function $g \in RH(\mathbb{D})$ of degree at most n that approximates f as closely as possible. We consider two versions of this problem. First we assume that f satisfies the interpolation condition (1), and we require g to satisfy the same interpolation conditions. Secondly, we relax the problem by removing the interpolation constraints.

Suppose that f has $\nu \leq n$ zeros in $\mathbb{D}$. Then $f = \pi f_0$, where f_0 is minimum-phase, and π is a polynomial of degree ν with zeros in $\mathbb{D}$. The approximant g can then be determined in two steps:

(i) Solve the quasi-convex optimization problem to find a pair $(p_0, q) \in \mathrm{Pol}(n-\nu) \times \mathrm{Pol}(n)$ that minimizes (18), as outlined in the Appendix. This yields optimal ϵ, p_0 and q. Set $p := \pi p_0$.

(ii) Solve the optimization problem (14) with $\rho = q/\tau$, as described in Theorem 5. Exchanging σ for ρ in (10) we solve the Vandermonde system

$$\beta(z_k) = q(z_k)w_k, \quad k = 0, 1, \ldots, n,$$

for the $\beta \in \mathrm{Pol}(n)$, which yields

$$g = \frac{\beta}{q} \tag{22}$$

and the bound (20), where $\tau(z) := \prod_{k=0}^{n}(1-\bar{z}_k z)$.

For the problem without interpolation condition, we replace step (ii) by one of the following steps.

(ii)$'$ Choose $z_0, z_1, \ldots, z_n$ arbitrarily, or as in Remark 2 below. This yields a solution (22) and a bound (20).

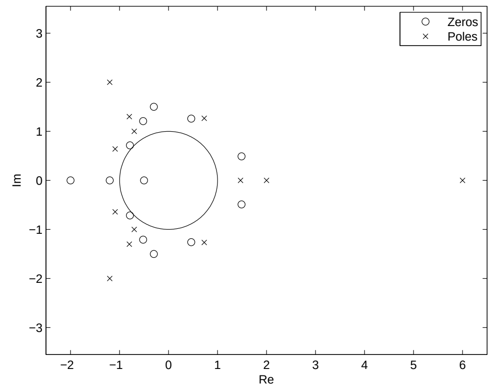

Fig. 2. Poles and zeros of f in Examples 1, 2, and 3.

(ii)$''$ Choose $z_0, z_1, \ldots, z_n$ so that τ is the outer (minimum-phase) factor of p. This yields a solution (22) and the bound (21) for the relative H_2 error.

Remark 2: If a bound on the weighted error $\|w(f-g)\|$ is desired in Step (ii)$'$, it is natural to choose τ so that $\frac{p}{\tau f}$ is as close to w as possible. This may be done by solving the convex optimization problem to find a $\tau \in \mathrm{Pol}(n)$ that minimizes

$$\left\|1 - \left|\frac{\tau f w}{p}\right|^2\right\|_\infty,$$

as in the Appendix. If instead we need a bound on the weighted relative error $\|w(f-g)/f\|$, we modify the optimization problem accordingly.

We apply these procedures to some numerical examples.

Example 1: Let

$$f(z) = \frac{b(z)}{a(z)}$$

be the stable system of order 13 given by

$$\begin{aligned}
b(z) \;&=\; 30z^{13} + 90z^{12} + 128.6z^{11} + 114.6z^{10} \\
&\quad -137.4z^9 - 322.3z^8 - 371.4z^7 + 10.8z^6 \\
&\quad +1005.8z^5 + 2428.7z^4 + 3967.0z^3 + 4189.7z^2 \\
&\quad +2800.6z + 726.2, \\
a(z) \;&=\; 4.0z^{13} - 13.4z^{12} - 44.2z^{11} - 144.5z^{10} \\
&\quad +83.5z^9 + 363.7z^8 + 791.4z^7 + 340.1z^6 \\
&\quad +770.7z^5 + 877.3z^4 - 93.6z^3 - 4767.8z^2 \\
&\quad -6349.3z - 4532.7.
\end{aligned}$$

This system has one minimum-phase zero. The poles and zeros are given in Figure 2.

Consider the problem to approximate f by a function g of degree six while preserving the values in the points $(z_0, z_1, \ldots, z_n) = (0, 0.3, 0.5, -0.1, -0.7, -0.3 \pm 0.3i)$.

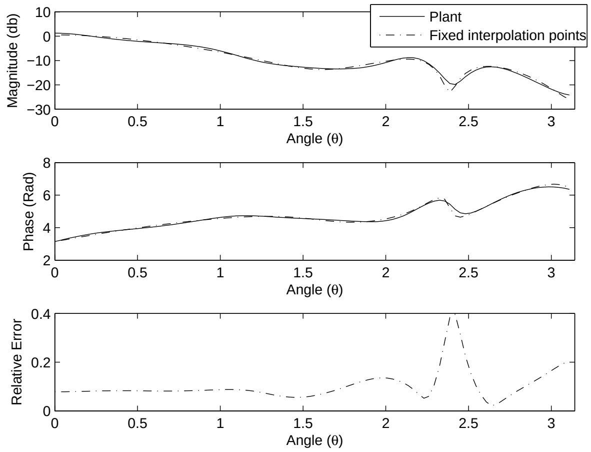

Fig. 3. Bode plots of f and g together with the relative error.

Such an interpolation condition occurs in certain applications.

Step (i) to solve the quasi-convex optimization problem to minimize (18) yields optimal ϵ, p and q, and Step (ii) the approximant g, the Bode plot of which is depicted in Figure 3 together with that of f. The third subplot in the picture shows the relative error

$$\left|\frac{f(e^{i\theta})-g(e^{i\theta})}{f(e^{i\theta})}\right| \text{ for } \theta\in[0,\pi].$$

It is important to note that the function g, which is guaranteed to be stable, satisfies the prespecified interpolation conditions and the error bound (20). Figure 3 shows that g matches f quite well.

Example 2: Next we approximate the function f in Example 1 without imposing any interpolation condition. For $n = 4$, 6 and 8 we determine an approximant g_n of degree n via Steps (i) and (ii)$''$. This approximant satisfies the relative error bound (21). Then we compare g_n to an approximant $\hat{f}_n$ of the same degree obtained by balanced truncation [19], [26].

Since balance truncation imposes a bound on the absolute, rather than the relative, error, it is reasonable to also compare it with the approximant h_n of degree n obtained by stochastically balanced truncation [25], [22], which comes with a relative error bound.

The respective Bode plots and relative errors for the three methods are depicted in Figures 4, 5, and 6. Stochastically balanced truncation gives the best approximation close to the valleys of the plant, and balanced truncation gives best approximation close to the peaks. The proposed method performs somewhere in between and has a more uniform relative error. In fact, as can be seen from Figure 5 and Figure 6, it is the method with the smallest relative L_∞-error for $n = 6$ and $n = 8$. As can be seen in the following tables, listing the relative and absolute errors of the three methods, the approximants of roughly the same quality.

Relative L_2 Error	Degree		
Approximation method	4	6	8
Proposed method	0.4736	0.0764	0.0194
Balanced truncation	0.4727	0.0785	0.0220
Stoch. Bal. truncation	0.7958	0.0656	0.0334

H_2 Error	Degree		
Approximation method	4	6	8
Proposed method	0.1918	0.0422	0.0100
Balanced truncation	0.0746	0.0451	0.0057
Stoch. Bal. truncation	0.3073	0.0506	0.0213

In the present example, the error bound (21) is quite conservative. In fact, the bound is $10.4735, 0.8765$, and 0.3994, for n equal to $4, 6$, and 8 respectively, which should be compared with the corresponding errors in the table. By comparison, the relative L_∞ bound on h_n is $3.9288, 0.3562$, and 0.0573 for n equal to $4, 6$, and 8 respectively, which is also conservative for $n = 4, 6$. Although these bounds are measured in different norms, it is still interesting to compare them. How to improve our bound will be subject to further studies.

In Figure 7 the approximant g from Example 1 is compared to g_6. The interpolation points for g_6 are chosen according to (ii)$''$, and the interpolation condition of g is prespecified. It can be seen from Figure 7 that g_6 matches f better than does g. This is because the interpolation points could be chosen freely for g_6.

Note that the problem of stable approximation could be approached directly by nonconvex optimization to find local optima by gradient methods (see e.g. [15] and references therein). If a sufficiently good starting point is provided then even the global optima could be reached. In this example it is possible to find, using such methods, approximations with relative errors 0.11, 0.0656, and 0.0105 of degree 4, 6, and 8, respectively.[1] These errors compares fovourable to all the above methods. However, our goal has been to provide an alternative framework based only on convex and quasi-convex optimization. An advantage with this approach is that the method does not rely on a good starting point for the algorithm which is often difficult to find. It will be subject to further research to investigate in which way optimal approximations of weights relate to optimal approximations of interpolants.

[1]The authors would like to thank professor Martine Olivi for providing us with this comparison.

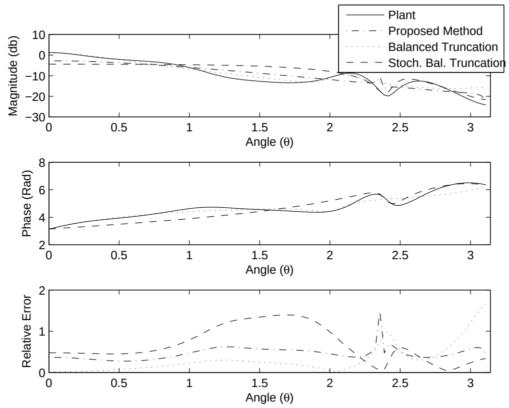

Fig. 4. *B*ode plot of f, g_4, $\hat{f}_4$, and h_4 together with the relative errors.

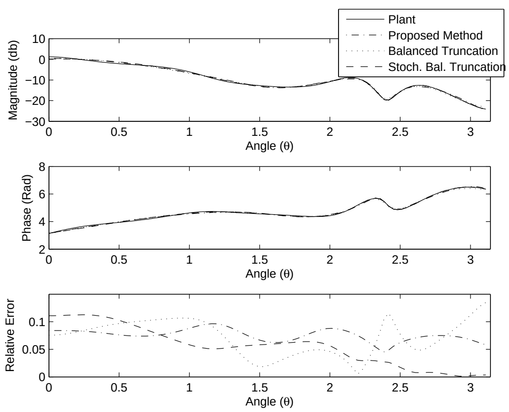

Fig. 5. *B*ode plot of f, g_6, $\hat{f}_6$, and h_6 together with the relative errors.

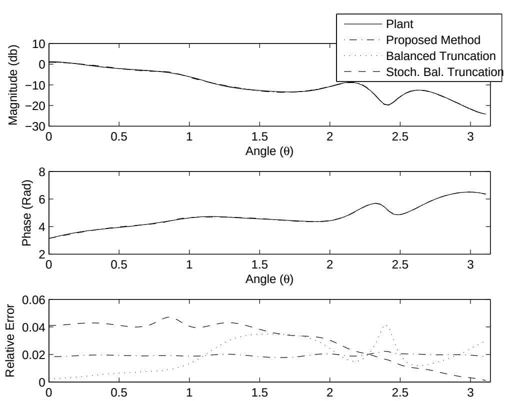

Fig. 6. *B*ode plot of f, g_8, $\hat{f}_8$, and h_8 together with the relative errors.

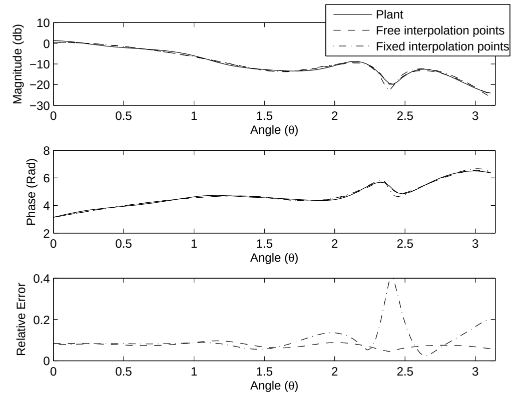

Fig. 7. *B*ode plot of f, g_6, and g together with the relative errors.

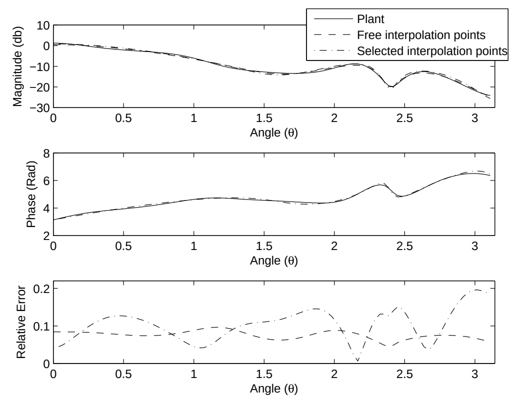

Fig. 8. *B*ode plot of f, g_6, and $\hat{g}_6$ together with the relative errors.

Example 3: We continue to approximate the function in Example 1, but this time we move the interpolation points to get a better fit in a selected frequency band. In computing g_6 the interpolation points were determined via (ii)$''$ to be

$$(0, -0.5, -0.8841, -0.0380 \pm 0.7221i, -0.7021 \pm 0.6488i),$$

thus yielding the weight $|p(e^{i\theta})/\tau(e^{i\theta})| = 1$ for $\theta \in [0, \pi]$. In order to get a better fit close to 1 (i.e. at $\theta = 0$) we replace the interpolation point -0.5 with the point 0.9, thus producing the weight

$$\left|\frac{p(e^{i\theta})}{\tau(e^{i\theta})}\right| = \left|\frac{1+0.5e^{i\theta}}{1-0.9e^{i\theta}}\right| \text{ for } \theta \in [0, \pi].$$

Denote by $\hat{g}_6$ the minimizer (14) corresponding to the interpolation points $(0, 0.9, -0.8841, -0.0380 \pm 0.7221i, -0.7021 \pm 0.6488i)$. The functions g_6 and $\hat{g}_6$ are depicted in Figure 8. In the selected region close to 1, $\hat{g}_6$ approximates the original system better than does g_6, but this is at the expense of the approximation in other regions of the unit circle.

Finally we show how the model reduction procedure may be applied for designing a low-degree controller.

Example 4 (Sensitivity shaping): In robust control, given a plant P, a controller is often designed by shaping the sensitivity function

$$S = \frac{1}{1 - PC},$$

where P and C are the transfer functions of the plant and the controller respectively. In fact, the design specifications may often be translated into conditions on the sensitivity function.

For internal stability of the closed loop system, the sensitivity function S needs to satisfy the following properties:

(i) S is analytic and bounded in $\mathbb{C}_+$,
(ii) $S(z_k) = 1$ whenever z_k is an unstable zero of P,
(iii) $S(p_k) = 0$ whenever p_k is an unstable pole of P.

Furthermore, in general we require that

(iv) S has low degree, and
(v) S satisfies additional design specifications.

The degree bound on S is important for several reasons. In fact, a low-degree sensitivity function results in a low-degree controller (see e.g., [17]), and, in some applications, the degree of the sensitivity function is important in its own right. A case in point is an autopilot, for which the feedback system itself is to be controlled. Conditions (i)-(iv) do not in general uniquely specify S, so the additional freedom can be utilized to satisfy additional design specifications (v).

To examplify the theory, we consider the sensitivity function $S = (1 - PC)^{-1}$ of the feedback system with plant

$$P(z) = \frac{1}{z - 2}.$$

Since P has one unstable pole at 2 and an unstable zero at ∞, we require that the sensitivity function satisfies

$$S(\infty) = 1 \text{ and } S(2) = 0.$$

We begin with a particular interpolant S_{ideal} without regard to any constraint on the degree, shown as a solid line in Figure 9. The function $f(z) = S(z^{-1})$ is analytic in $\mathbb{D}$, and satisfies

$$f(0) = 1 \text{ and } f(1/2) = 0.$$

By using the computational procedure in the beginning of the section we find degree r approximations f_r of $f_{\text{ideal}}(z) = S_{\text{ideal}}(z^{-1})$. Then the sensitivity functions S_r are obtained from $S(z) = f(z^{-1})$.

We compute S_r for $r = 1, 2, 3$ and display their magnitudes in Figure 9. It is interesting to note that even though S_{id} is infinite-dimensional it is possible to find satisfactaory low-dimensional approximants.

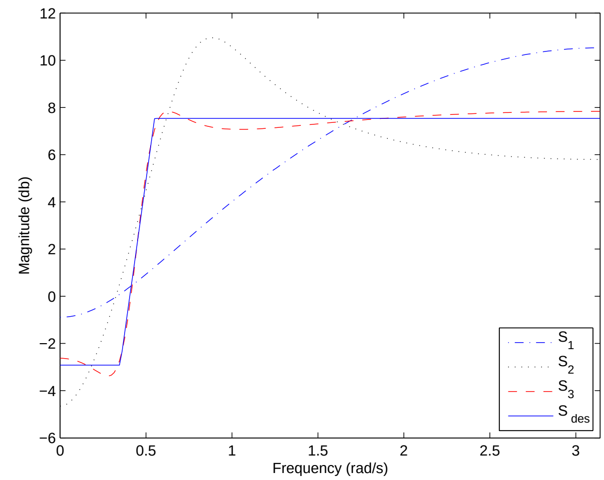

Fig. 9. Approximations of degree $1, 2,$ and 3

Example 5: In [9], the problem of shaping the sensitivity function of a flexible beam with transfer function

$$P(s) = \frac{-6.4750s^2 + 4.0302s + 175.770}{s(5s^3 + 3.5682 + 139.5021s + 0.09290)}$$

is considered, and a controller is sought so that the sensitivity function is close to

$$S_{id} = \frac{s(s + 1.2)}{s^2 + 1.2s + 1},$$

whose Bode plot is depicted in Figure 10. The plant P has an unstable zero in 5.5308, a pole at 0 and has relative degree 2. For the controller to be strictly proper and the closed loop system to be internally stable, the interpolation condition

$$\begin{aligned} S(5.5308) &= S(\infty) = 1, \\ S(0) &= S(\infty)' = S(\infty)'' = 0, \end{aligned}$$

needs to be satisfied.

In order to apply our theory as presented in this paper, we first transform the domain of the problem from $\mathbb{C}_+$ to $\mathbb{D}$, using the bilinear transformation

$$s \rightarrow z = \frac{s_0 - s}{s_0 + s}, \text{ where } s_0 = 3.1.$$

The constant $s_0 = 3.1$ is chosen the corresponding bilinear transformation maps the area of interest, $0.1i$ to $100i$, onto a large part of the unit circle. Choosing s_0 too small or too large might cause numerical problems. This yields

$$f_{id}(z) = S_{id}\left(s_0 \frac{1 - z}{1 + z}\right),$$

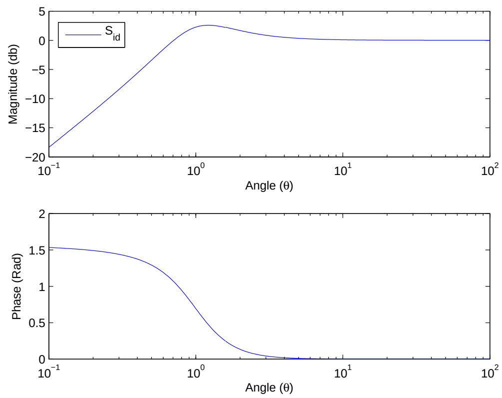

Fig. 10. Bode plot of S_{id} and S

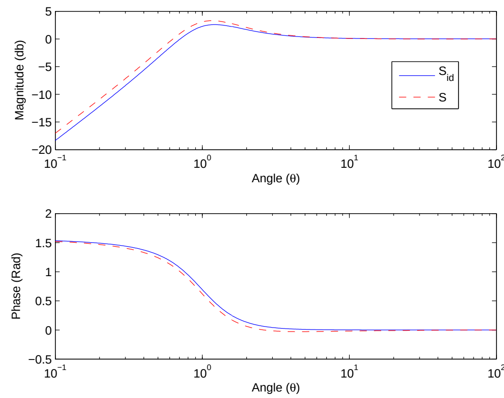

Fig. 11. Bode plot of S_{id} and S

and the problem is then to find a stable function g that is close to $f_{id}(z)$ and which satisfies the constraints

$$\begin{aligned} g(-0.2816) &= g(-1) = 1, \\ g(1) &= g(-1)' = g(-1)'' = 0. \end{aligned} \tag{23}$$

However, f_{id} does not satisfy the constraints (23), and therefore the method in Section V does not directly apply. Instead we would like to find an approximation f of f_{id} which satisfies the interpolation constraints, and then apply the degree reduction method on f.

Note that it is impossible to obtain an analytic function f which simultaneously satisfies the interpolation condition (23) and the criterion $|f(z)| \leq |f_{id}(z)|$ for $z \in \mathbb{T}$. If such a function f did exist, then $B := f/f_{id}$ would be analytic in $\mathbb{T}$ and bounded by one on $\mathbb{T}$. However,

$$B(-0.2816) = f(-0.2816)/f_{id}(-0.2816) = 1.0269 > 1,$$

and hence B violates the maximum principle. Therefore we need to be content with a function f which satisfies $|f(z)| \leq |f_{id}(z)|(1+\epsilon)$ for $z \in \mathbb{T}$ with some $\epsilon > 0.0269$.

If all the interpolation points of f were in $\mathbb{D}$, a straightforward method would be to take f as the minimizer of

$$\left\| \frac{f(z)}{f_{id}(z)} \right\|_\infty \text{ subject to (23).}$$

Then we would have $f = f_{id} B \alpha$, where B is a Blaschke product and $\alpha > 0$. But, since there are interpolation points on the boundary, a slightly larger region of analyticity need to be considered.

Note that f_{id} is analytic in $(1+\delta)\mathbb{D} := \{(1+\delta)z : |z| < 1\}$ for $0 < \delta < 0.44$, and let f be the function that minimizes

$$\left\| \frac{f(z)}{f_{id}(z)} \right\|_{H_\infty((1+\delta)\mathbb{D})}$$

subject to f satisfying the constraints (23). Now, for any $\epsilon > 0.0269$ one can find a $\delta > 0$ so that $\left\| \frac{f(z)}{f_{id}(z)} \right\|_\infty \leq 1+\epsilon$. We choose $\epsilon = 0.05$, and for this ϵ, $\delta = 0.05$ works.

Then the function f satisfies $|f(z)| \leq 1.05|f_{id}(z)|$ for $z \in \mathbb{T}$, and, since (23) holds for f it is possible to follow the steps (i) and (ii) to reduce the degree of f to 4. That is, let $(\rho, \sigma) \in \mathcal{K}_0 \times W_f$ be the minimizer of

$$\left\| 1 - \left| \frac{\rho}{\sigma} \right|^2 \right\|_\infty,$$

and let g be the unique function satisfying (23) and $\rho g \in \mathcal{K}$. Finally we transform the domain back to the continuous-time setting via

$$z \to s = s_0 \frac{1-z}{1+z},$$

whcih gives $S(s) = g\left(\frac{s_0 - s}{s_0 + s}\right)$ as depicted in Figure 11.

Note that since there are interpolation points on the boundary, the relative H_2 bound is not meaningful. In fact, σ has poles in -1 that are not cancelled by zeros of f, and hence the right hand side of

$$\|\sigma(f-g)\|^2 \leq \frac{4\epsilon}{1-\epsilon} \|\sigma f\|^2.$$

will be infinite, rendering the inequality trivial. How to deal with interpolation points on the boundary in a more rigorous way will be the subject of further research.

It is worth noting that if the main concern is a low order controller, one can consider a larger class of sensitivity functions with a possibility of better design. For clarity of presentation we will consider a discrete-time plant P. Briefly, we recall from [17] that

$$\deg C \leq \deg P + \deg S - n_p - n_z$$

where n_p and n_z are the number of unstable zeros and poles respectively of the plant P. Since the theory guarantees that $\deg S \leq n_p + n_z - 1$, the degree of the controller is less than the degree of the plant P. We then factor the transfer function of the plant into a stable and an unstable part as

$$P = \frac{\beta_u \beta_s}{\alpha_u \alpha_s},$$

where β_u and α_u have roots in $\mathbb{D}$, and β_s and α_s have roots in $\mathbb{D}^C$. The idea is to use our knowledge about the stable part of the plant to construct a larger class of sensitivity functions for which the controller order is the same. Let

$$\mathcal{K}_{\alpha_s} = \left\{ \sigma = \frac{b}{\tau \alpha_s}, b \in \mathrm{Pol}(n + \deg \alpha_s), \sigma \text{ outer} \right\},$$

where

$$\tau(z) = \prod_{k=1}^{n_z} (1 - \bar{z}_k z) \prod_{k=1}^{n_p} (1 - \bar{z}_p z).$$

Now for any $\sigma \in \mathcal{K}_{\alpha_s}$ the minimizer of

$$\min \|\sigma S\| \text{ s. t. } \begin{cases} S(z_k) = 1, & k = 0, \ldots, n_z \\ S(p_k) = 0, & k = 0, \ldots, n_p, \end{cases}$$

is of the form $S = \frac{\alpha_s a}{b}$, where $a \in \mathrm{Pol}(n)$. Due to the interpolation constraints we have

$$\alpha_u | a, \quad \beta_u | (\alpha_s a - b), \tag{24}$$

and hence

$$C = \frac{S - 1}{PS} = \frac{\alpha_u(\alpha_s a - b)}{\beta_u \beta_s a} = \frac{\frac{\alpha_s a - b}{\beta_u}}{\frac{\beta_s a}{\alpha_u}}.$$

In view of (24), $\frac{\alpha_s a - b}{\beta_u}$ and $\frac{\beta_s a}{\alpha_u}$ are polynomials, and since $\deg \alpha_u + \deg \beta_u = n$, we have

$$\deg \frac{\alpha_s a - b}{\beta_u} \le n + \deg \alpha_s - \deg \beta_u = \deg \alpha_s + \deg \alpha_u$$

and

$$\deg \frac{\beta_s a}{\alpha_u} \le n + \deg \beta_s - \deg \alpha_u = \deg \beta_s + \deg \beta_u.$$

This shows that any choice of σ in the class $\mathcal{K}_{\alpha_s}$ will produce a controller of a degree less than the degree of the plant. By utilizing the stable part of the plant, we have shown that choosing sensitivity functions from a larger class will not increase the degree of the controller.

VIII. Conclusions and further work

In this paper, we propose a method for degree reduction of stable systems. The method is based on weighted H_2 minimization under interpolation constraints. By choosing weights appropriately, the minimizer will both be of low degree and match the original system. This gives a model reduction procedure for the case that both the original system and the degree-reduced system satisfy prespecified interpolation conditions (Section V). In the case where no such interpolation conditions are required, we provide a systematic procedure which utilizes the extra freedom of choosing the interpolation points (Section VI). The various versions of the model reduction procedure are then demonstrated on a simple example, and finally the method is applied to a control design example from [9].

The study of the H_2 minimization problem is motivated by the relation between the H_2 norm and the entropy functional used in bounded interpolation. Therefore, new concepts derived in this framework are useful for understanding entropy minimization. In fact, both the degree reduction methods proposed in this paper easily generalize to the bounded case; see [12] for the method which preserves interpolation conditions. We are currently working on similar bounds for the positive real case; also, see [10].

IX. Acknowledgements

The authors would like to thank professor Tryphon Georgiou for many interesting discussions and for suggesting the reference [16]. Some of the ideas that lead to this paper originated in the join work with professor Georgiou in [12].

Appendix

A quasi-convex optimization problem is an optimization problem for which each sublevel set is convex. The optimization problem to minimize (19), where p and q are polynomials of fixed degree is quasi-convex. For simplicity, we assume that f is real and hence that p and q are real as well.

As a first step, consider the *feasibility problem* of finding a pair (p, q) of polynomials satisfying

$$\left\| 1 - \left| \frac{qf}{p} \right|^2 \right\|_\infty \le \epsilon \tag{25}$$

for a given ϵ, or, equivalently,

$$-\epsilon |p(e^{i\theta})|^2 \le |p(e^{i\theta})|^2 - |q(e^{i\theta}) f(e^{i\theta})|^2 \le \epsilon |p(e^{i\theta})|^2$$

for all $\theta \in [-\pi, \pi]$. Since $|p|^2$ and $|q|^2$ are pseudo-polynomials, they have representations

$$\begin{aligned} |p(e^{i\theta})|^2 &= 1 + \sum_{k=1}^{n_p} p_k \cos(k\theta), \\ |q(e^{i\theta})|^2 &= \sum_{k=0}^{n_q} q_k \cos(k\theta), \end{aligned}$$

where n_p and n_q are the degree bounds on p and q respectively, and the first coefficient in $|p|^2$ is chosen to be one without loss of generality. Hence (25) is equivalent to

$$-1-\epsilon \leq (1+\epsilon)\sum_{k=1}^{n_p} p_k \cos k\theta - |f(e^{i\theta})|^2 \sum_{k=0}^{n_q} q_k \cos k\theta,$$

$$1-\epsilon \leq (\epsilon-1)\sum_{k=1}^{n_p} p_k \cos k\theta + |f(e^{i\theta})|^2 \sum_{k=0}^{n_q} q_k \cos k\theta,$$

for all $\theta \in [-\pi, \pi]$. There is also a requirement on $1+\sum_{k=1}^{n_p} p_k \cos(k\theta)$ and $\sum_{k=0}^{n_q} q_k \cos(k\theta)$ to be positive. However, if $\epsilon \in (0,1)$, then the above constraints will imply positivity. The set of $p_1, p_2, \ldots, p_{n_p}, q_0, q_1, \ldots, q_{n_q}$ satisfying this infinite number of linear constraints is convex.

The most straightforward way to solve this feasibility problem is to relax the infinite number of constraints to a finite grid, which is dense enough to yield an appropriate solution. Here one must be carefully to check the positivity of $1+\sum_{k=1}^{n_p} p_k \cos(k\theta)$ and $\sum_{k=0}^{n_q} q_k \cos(k\theta)$ in the regions between the grid points. Another method is the Ellipsoid Algorithm, described in detail in [4].

Minimizing (19) then amounts to finding the smallest ϵ for which the feasibility problem has a solution. This can be done by the the bisection algorithm, as described in [4]. Note that for $\epsilon = 1$, the trivial solution $q = 0$ is always feasible.

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